A HOMOTOPY METHOD FOR SOLVING THE HORIZONTAL LINEAR COMPLEMENTARITY PROBLEM

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Abstract. The $P(\tau, \alpha, \beta)-$pair defined in this paper is a class of matrix pair which is broad enough to include P^*-matrix as special case. We construct a combined homotopy equation for the horizontal linear complementarity problem, prove the existence, boundedness and the convergence of the homotopy path, which is from any interior point to the solution of the problem, under a condition that the matrix pair is $P(\tau, \alpha, \beta)$ pair. Numerical examples show that this method is feasible and effective.

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1. Introduction

In this paper, we will study the following horizontal linear complementarity problem(HLCP): finding a vector $x \in R^n$ such that $Mx - Ny + q = 0,\ x \geq 0,\ y \geq 0$ and $x^T y = 0$, where $M,\ N$ are $n \times n$ matrices, $q \in R^n$ is a vector. If $N = E$, where E is identity matrix, the horizontal linear complementarity problem reduces to the linear complementarity problem, that is, to find $x \geq 0$, such that $y = Mx + q \geq 0$ and $x^T y = 0$.

The horizontal linear complementarity problem(HLCP) is a class of impotent complementarity problems. There are many methods to

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solve horizontal linear complementarity problem, Roman et al. [1] have generalized the ω-properties. zhang et al. [2] have given the s-type error bound for the horizontal linear complementarity problem. Frede et al. [3] have introduced the infeasible path following algorithm for the horizontal complementarity problem. Monteiro et al. [4] have studied the monotone horizontal linear complementarity problem. Stoer [5] has given high order long-step method for monotone horizontal linear complementarity problem. Liu et al. [6] and Filiz et al. [7] have given the iteration complexity of a higher order corrector-predictor interior-point method for the sufficient horizontal linear complementarity problem. However, all those methods as mentioned above need the condition that the solution set is nonempty and few attempts have been done on using homotopy method to solve the horizontal linear complementarity problem. The objective of this paper is to construct a homotopy equation for the horizontal linear complementarity problem, give the solvability of the horizontal linear complementarity problem with $P(\tau, \alpha, \beta)$ matrix pair.

Throughout the paper, all vectors are column vectors, the vector $(x^T, y^T)^T \in R^n \times R^n$ is usually abbreviated by $(x,\ y)$ and superscript T denotes the transpose of a vector. For any $x \in R^n$, we denote by $\|x\|$ the Euclidean norm of x, by x_i the ith component of x, we denote by R^n_+ (respectively, R^n_{++}) the space of $n-$dimensional real vectors with nonnegative components(respectively, positive components). When $x \in R^n_+$(respectively, R^n_{++}), we also write $x \geq 0$(respectively, $x > 0$) for simplicity. $H'(\omega)$ denotes the Jacobian matrix of the $H(\omega)$. Let $L = \{1, 2, \cdots, n\}$.

The organization of the paper is as follows. In section 2, we introduce some definitions and some basic preliminaries for the horizontal linear complementarity problem (HLCP) that will be utilized in the paper. In Section 3, we construct a homotopy equation for the HLCP and obtain the solvability of this problem. Some numerical examples are given to show the effectiveness and feasibility of this method.

2. Preliminaries and Definitions

We first introduce three lemmas from differential topology which will be used in the following discussions.

Let $U \subseteq R^n$ be an open set and let $\phi : U \to R^p$ be a C^α ($\alpha > \max\{0, n-p\}$) mapping. We say that $y \in R^p$ is a regular value for ϕ, if

$$Range\left[\frac{\partial \phi(x)}{\partial x}\right] = R^p, \ \forall x \in \phi^{-1}(y).$$

Lemma 2.1. *(see[9]) (Parameterized Sard Theorem on smooth manifold) Let $V \subset R^n$, $U \subset R^m$ be open sets, and let $\phi : V \times U \to R^k$ be a C^α mapping, where $\alpha > \max\{0, m-k\}$. If $0 \in R^k$ is a regular value of ϕ, then for almost all $a \in V$, 0 is a regular value of $\phi_a = \phi(a, \cdot)$.*

Lemma 2.2. *(see[10]) (The inverse image theorem) Let $\phi : U \subset R^n \to R^p$ be a C^α ($\alpha > \max\{0, n-p\}$) mapping. If $0 \in R^p$ is a regular value of ϕ, then $\phi^{-1}(0)$ consists of some $(n-p)$-dimensional C^α manifolds.*

Lemma 2.3. *(see[10]) (Classification theorem of one-dimensional smooth manifold) One-dimensional smooth manifold is diffeomorphic to a unit circle or a unit interval.*

The following matrix pair has been extensively used in the literature to ensure the solvability of the HLCP.

Definition 2.4. (see[2,5])The pair$(M,\ N)$ is said P pair, if for any $u, v \in R^n,\ u \neq 0,\ Mu - Nv = 0$ implies

$$u^T v \geq 0.$$

This matrix pair is called the $X-$column monotonicity with respect to R^n_+ by Jianzhong Zhang, Naihua Xiu in [2]. This matrix pair is also called monotone by Josef Stoer in [5].

Definition 2.5. (see[6,7]) The pair(M,N) is said $P_*(\kappa)$ pair, if for some scalar $\kappa \geq 0$, $Mu - Nv = 0$ implies

$$(1+4\kappa)\sum_{i\in I_+(u,v)} u_iv_i + \sum_{i\in I_-(u,v)} u_iv_i \geq 0$$

for any $u, v \in R^n$, where

$$I_+(u,v) = \{i|u_iv_i > 0\}, I_-(u,v) = \{i|u_iv_i \leq 0\}.$$

We say the horizontal linear complementarity problem is a $P_*(\kappa)$ HLCP, if the pair $(M,\ N)$ is a $P_*(\kappa)$ pair. In the case $N = E$, $(M,\ E)$ is a $P_*(\kappa)$ pair if and only if M is a $P_*(\kappa)$ matrix, that is

$$(1+4\kappa)\sum_{i\in I_+(u)} u_i(Mu)_i + \sum_{i\in I_-(u)} u_i(Mu)_i \geq 0,$$

for $u \in R^n$, $I_+(u) = \{i \in L|u_i(Mu)_i > 0\}, I_-(u) = \{i \in L|u_i(Mu)_i \leq 0\}$. If $(M,\ N)$ belongs to the class $P_* = \bigcup_{\kappa\geq 0} P_*(\kappa)$, then we say $(M,\ N)$ is a P_* pair and HLCP is said a P_* HLCP.

Example 2.1 Let

$$M = \begin{pmatrix} 10 & -6 & 0 \\ -5 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ N = \begin{pmatrix} 2 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

hence, we have

$$Mu = Nv \Leftrightarrow \begin{cases} 10u_1 - 6u_2 = 2v_1 + 2v_2 \\ -5u_1 - 3u_2 = v_1 - v_2 \\ u_3 = v_3 \end{cases}$$

and

$$u_1v_1 = -3u_1u_2,\ u_2v_2 = 5u_1u_2,\ u_3v_3 = u_3^2,$$

$(M,\ N)$ is a P_* pair with $\tau = 1$, but $(M,\ N)$ is not a P pair. In fact, let $(0,3,0)^T$, we have $(v_1,v_2,v_3)^T = (-9,0,0)^T$, $\max_{1\leq i\leq 3} u_iv_i = 0$.

We combine the idea of the reference [8] and definition 2.5 to give the following definitions 2.6 and 2.7.

Definition 2.6. The pair$(M,\ N)$ is said $P(\tau,\alpha,\beta)$ pair, if for some scalar $\tau \geq 0,\ \alpha > 0,\ 0 \leq \beta < 1$, $Mu - Nv = 0$ implies

$$(1+\tau)\sum_{i\in I_+(u,v)} u_iv_i + \sum_{i\in I_-(u,v)} u_iv_i \geq -\alpha\|u\|^\beta$$

for any $u, v \in R^n$, where

$$I_+(u,v) = \{i|Mu - Nv = 0,\ u_iv_i > 0\},$$
$$I_-(u,v) = \{i|Mu - Nv = 0,\ u_iv_i \leq 0\}.$$

Clearly, a P_*-pair must be a $P(\tau,\alpha,\beta)$ pair.

Lemma 2.7. *If the pair $(M,\ N)$ is a P-pair, then $(M,\ N)$ is a P_* pair.*

Proof. Let $B = \{(u,v) \in R^{n\times n} | \|(u,v)\| = 1\}$, if the pair $(M,\ N)$ is a P-pair, from Definition 2.4, $\max\limits_{1\leq i\leq n} u_iv_i > 0$ for any $u,\ v$ satisfying $Mu - Nv = 0$. We define a multivariate function:

$$\varphi(u,v): R^{n\times n} \to R, \varphi(u,v) = \frac{\sum\limits_{i=1}^{n} u_iv_i}{\sum\limits_{i\in I_+(u,v)} u_iv_i}\ (\sum_{i\in I_+(u,v)} u_iv_i > 0),$$

hence, $\varphi(u,v)$ is a continuous function, for any $(u,\ v) \neq 0$, there exists a constant $\tau \geq 0$, such that, for any $(u,\ v) \in B$,

$$-\tau \leq |\varphi(u,v)| \leq \tau,$$

i.e. for any $(u,\ v) \in B$,

$$\sum_{i=1}^{n} u_iv_i \geq -\tau \sum_{i\in I_+(u,v)} u_iv_i.$$

Rearranging the terms

$$(1+\tau)\sum_{i\in I_+(u,v)} u_iv_i + \sum_{i\in I_-(u,v)} u_iv_i \geq 0.$$

For any $Mu - Nv = 0$, $0 \neq (u,\ v) \in R^{n\times n}$, let $\tilde{u} = \frac{u}{\|(u,v)\|}, \tilde{v} = \frac{v}{\|(u,v)\|}, (\tilde{u},\tilde{v}) \in B$, so we have

$$(1+\tau)\sum_{i\in I_+(\tilde{u},\tilde{v})} \tilde{u}_i\tilde{v}_i + \sum_{i\in I_-(\tilde{u},\tilde{v})} \tilde{u}_i\tilde{v}_i \geq 0.$$

Multiplying above inequality both side by $\|(u,v)\|^2$, one has

$$(1+\tau)\sum_{i\in I_+(\tilde{u},\tilde{v})}\tilde{u}_i\tilde{v}_i\|(u,v)\|^2+\sum_{i\in I_-(\tilde{u},\tilde{v})}\tilde{u}_i\tilde{v}_i\|(u,v)\|^2\geq 0.$$

Noticing

$$I_+(\tilde{u},\tilde{v})=I_+(u,v), I_-(\tilde{u},\tilde{v})=I_-(u,v),$$

hence, we have

$$(1+\tau)\sum_{i\in I_+(u,v)}u_iv_i+\sum_{i\in I_-(u,v)}u_iv_i\geq 0.$$

□

Definition 2.8. A matrix M is called a $P(\tau,\alpha,\beta)$ matrix, if for any $x\in R^n$, there exists positive scalars $\tau\geq 0,\ \alpha>0,\ 0\leq\beta\leq 1$, such that

$$(1+\tau)\sum_{i\in I_+(x)}x_i(Mx)_i+\sum_{i\in I_-(x)}x_i(Mx)_i\geq -\alpha\|x\|^\beta,$$

where $I_+(x)=\{i|x_i(Mx)_i>0\}, I_-(x)=\{i|x_i(Mx)_i\leq 0\}$.

Remark 2.9. If N is nonsingular, the pair (M,N) is a $P(\tau,\alpha,\beta)$ pair if and only if $N^{-1}M$ is a $P(\tau,\alpha,\beta)$ matrix.

We make the following assumptions for the horizontal linear complementarity problem:

H1 There exists $\bar{x}>0,\bar{y}>0$ such that $M\bar{x}-N\bar{y}+q=0$;

H2 The pair $(M,\ N)$ is full row rank, without loss of generality, we assume that N is invertible.

3. Main Results

For any $x^{(0)}\in R^n_{++}, y^{(0)}\in R^n_{++}$, let

$$\omega=(x,y), X=diag(x)=\begin{pmatrix} x_1 & 0 & \cdots & 0\\ 0 & x_2 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & x_n\end{pmatrix}.$$

We construct a homotopy equation as follows:

$$H(\omega, \omega^{(0)}, \mu) = \begin{pmatrix} (1-\mu)(Mx+q) - Ny + \mu N y^{(0)} \\ Xy - \mu X^{(0)} y^{(0)} \end{pmatrix} = 0. \tag{1}$$

Where $\omega = (x, y),\ \omega^{(0)} = (x^{(0)}, y^{(0)})$.

When $\mu = 1$, equation (1) becomes

$$\begin{pmatrix} N(-y + y^{(0)}) \\ Xy - X^{(0)} y^{(0)} \end{pmatrix} = 0$$

and it has a unique solution. When $\mu = 0$, equation (1) becomes

$$\begin{pmatrix} Mx - Ny + q \\ Xy \end{pmatrix} = 0.$$

Obviously, $H(\omega^{(0)}, \omega^{(0)}, 1) = 0$. For given $\omega^{(0)} \in R^n_{++} \times R^n_{++}$, we also write $H(\omega, \omega^{(0)}, \mu)$ in (1) as $H_{\omega^{(0)}}(\omega, \mu)$. Let

$$H^{-1}_{\omega^{(0)}}(0) = \{(\omega, \mu) \in R^n_+ \times R^n_+ \times (0, 1] \mid H(\omega, \omega^{(0)}, \mu) = 0\}.$$

Lemma 3.1. *If H1, H2 hold, then for almost all $\omega^{(0)} \in R^n_{++} \times R^n_{++}$, 0 is a regular value of H. The homotopy equation (1) generates a smooth curve $\Gamma_{\omega^{(0)}}$ starting from $(x^{(0)}, y^{(0)}, 1)$.*

Proof. We use $H'(\omega, \omega^{(0)}, \mu)$ to represent the Jacobian matrix of H. Then

$$H'(\omega, \omega^{(0)}, \mu) = \left(\frac{\partial H}{\partial \omega}, \frac{\partial H}{\partial \omega^{(0)}}, \frac{\partial H}{\partial \mu} \right),$$

For $\omega^{(0)} \in R^n_{++} \times R^n_{++}$, we have

$$\frac{\partial H}{\partial \omega^{(0)}} = \begin{pmatrix} 0 & \mu N \\ -\mu diag(y^{(0)}) & -\mu X^{(0)} \end{pmatrix},$$

where E is the identity matrix, then by H2 we have

$$det\left(\frac{\partial H}{\partial \omega^{(0)}} \right) = (-1)^{2n} \mu^{2n} \det(N) \prod_{i=1}^{n} y_i^{(0)} \neq 0, (\mu \in (0, 1]).$$

Therefore, $H'(\omega, \omega^{(0)}, \mu)$ is a full row rank matrix. By Lemma 2.1, we know that 0 is a regular value of $H(\omega, \omega^{(0)}, \mu)$ and by Lemma 2.2, $H^{-1}_{\omega^{(0)}}(0)$ consists of some smooth curves. The equation $H(\omega^{(0)}, \omega^{(0)}, 1) =$

0 implies that there exists a smooth curve $\Gamma_{\omega^{(0)}}$ starting from $(\omega^{(0)}, 1)$. □

Lemma 3.2. *For a given $\omega^{(0)} \in R^n_{++} \times R^n_{++}$, if 0 is a regular value of H and the pair (M, N) is a $P(\tau, \alpha, \beta)$ pair, then $\Gamma_{\omega^{(0)}}$ is a bounded curve in $R^n_+ \times R^n_+ \times (0, 1]$.*

Proof. From (1), it is easy to see that $\Gamma_{\omega^{(0)}} \subset R^n_+ \times R^n_+ \times (0, 1]$. If $\Gamma_{\omega^{(0)}}$ is an unbounded curve, then there exists a sequence of points $(x^{(k)}, y^{(k)}, \mu_k) \in \Gamma_{\omega^{(0)}}$, such that $\|(x^{(k)}, y^{(k)}, \mu_k)\| \to \infty$ as $k \to \infty$. If $\|x^{(k)}\| \to \infty$, from equation (1), we have

$$x_i^{(k)} y_i^{(k)} - \mu_k x_i^{(0)} y_i^{(0)} = 0, \quad i \in L. \tag{2}$$

It follows immediately from (2) that $x_i^{(k)} > 0, y_i^{(k)} > 0, \quad i \in L$. By the first equality of the homotopy equation (1), we have

$$-Ny^{(k)} + (1 - \mu_k)(Mx^{(k)} + q) + \mu_k N y^{(0)} = 0. \tag{3}$$

By assumption H1, one has

$$(1 - \mu_k)(-N\bar{y} + M\bar{x} + q) = 0.$$

Combining (3) and above equation, we have

$$N(y^{(k)} - (1 - \mu_k)\bar{y} - \mu_k y^{(0)}) - (1 - \mu_k)M(x^{(k)} - \bar{x}) = 0. \tag{4}$$

Let

$$u^{(k)} = x^{(k)} - \bar{x}, v^{(k)} = \frac{1}{1 - \mu_k}(y^{(k)} - (1 - \mu_k)\bar{y} - \mu_k y^{(0)}),$$

then by (4), we have

$$Mu^{(k)} - Nv^{(k)} = 0,$$

$$u_i^{(k)} v_i^{(k)} = \frac{1}{1 - \mu_k}(x_i^{(k)} - \bar{x}_i)[y_i^{(k)} - (1 - \mu_k)\bar{y}_i - \mu_k y_i^{(0)}]$$

and

$$\begin{aligned} &(x_i^{(k)} - \bar{x}_i)[y_i^{(k)} - (1 - \mu_k)\bar{y}_i - \mu_k y_i^{(0)}] \\ &= \mu_k x_i^{(0)} y_i^{(0)} - [(1 - \mu_k)\bar{y}_i + \mu_k y_i^{(0)}]x_i^{(k)} \\ &-\bar{x}_i y_i^{(k)} + \bar{x}_i[(1 - \mu_k)\bar{y}_i + \mu_k y_i^{(0)}]. \end{aligned} \tag{5}$$

It follows from (5) that, for any $i \in B = \{i|x_i^{(k)} \to \infty\}$, one has

$$(x_i^{(k)} - \bar{x}_i)[y_i^{(k)} - (1-\mu_k)\bar{y}_i - \mu_k y_i^{(0)}] \to -\infty, \tag{6}$$

as $i \notin B$,

$$\begin{aligned} &(x_i^{(k)} - \bar{x}_i)[y_i^{(k)} - (1-\mu_k)\bar{y}_i - \mu_k y_i^{(0)}] \\ &\le \mu_k x_i^{(0)} y_i^{(0)} + \bar{x}_i[(1-\mu_k)\bar{y}_i + \mu_k y_i^{(0)}] \\ &\le x_i^{(0)} y_i^{(0)} + \bar{x}_i(\bar{y}_i + y_i^{(0)}). \end{aligned} \tag{7}$$

There exists a subsequence of $\{x^{(k)}\}$, also denoted by $\{x^{(k)}\}$, such that, there exists some index s and p, for all sufficiently large k, one has

$$u_s^{(k)} = x_s^{(k)} - \bar{x}_s = \max_{1\le i\le n}(x_i^{(k)} - \bar{x}_i) = \max_{1\le i\le n} u_i^{(k)},$$

$$u_p^{(k)} v_p^{(k)} = \max_{1\le i\le n} u_i^{(k)} v_i^{(k)} = \max_{1\le i\le n} \frac{1}{1-\mu_k}(x_i^{(k)} - \bar{x}_i)[y_i^{(k)} - (1-\mu_k)\bar{y}_i - \mu_k y_i^{(0)}].$$

Combined (6) and (7) and $(M,\ N)$ is a $P(\tau, \alpha, \beta)$ pair, for sufficiently large k, we have

$$\begin{aligned} u_s^{(k)} v_s^{(k)} &\ge \min_{1\le i\le n} \tfrac{1}{1-\mu_k}(x_i^{(k)} - \bar{x}_i)[y_i^{(k)} - (1-\mu_k)\bar{y}_i - \mu_k y_i^{(0)}] \\ &\ge -\tau \tfrac{1}{1-\mu_k} \max_{1\le i\le n}(x_i^{(k)} - \bar{x}_i)[y_i^{(k)} - (1-\mu_k)\bar{y}_i - \mu_k y_i^{(0)}] - \alpha\|u^{(k)}\|^\beta \\ &\ge -\tau \tfrac{1}{1-\mu_k}[x_p^{(0)} y_p^{(0)} + \bar{x}_p(\bar{y}_p + y_p^{(0)})] - \alpha\|x^{(k)} - \bar{x}\|^\beta. \end{aligned}$$

So we have

$$\begin{aligned} v_s^{(k)} &\ge -\tau \frac{u_p^{(k)} v_p^{(k)}}{u_s^{(k)}} - \frac{\alpha\|u^{(k)}\|^\beta}{u_s^{(k)}} \\ &\ge -\frac{\tau \frac{1}{1-\mu_k}[x_p^{(0)} y_p^{(0)} + \bar{x}_p(\bar{y}_p + y_p^{(0)})]}{x_s^{(k)} - \bar{x}_s} - \alpha \frac{\|x^{(k)} - \bar{x}\|^\beta}{x_s^{(k)} - \bar{x}_s}. \end{aligned}$$

Multiplying above inequality both side by $1-\mu_k$ and noticing

$$v^{(k)} = \frac{1}{1-\mu_k}(y^{(k)} - (1-\mu_k)\bar{y} - \mu_k y^{(0)}),$$

we have

$$
\begin{aligned}
y_s^{(k)} - (1-\mu_k)\bar{y}_s - \mu_k y_s^{(0)} \quad &\geq \quad -\tau \tfrac{x_p^{(0)} y_p^{(0)} + \bar{x}_p(\bar{y}_p + y_p^{(0)})}{x_s^{(k)} - \bar{x}_s} \\
&- \quad \alpha(1-\mu_k)\tfrac{\|x^{(k)} - \bar{x}\|^\beta}{x_s^{(k)} - \bar{x}_s},
\end{aligned} \tag{8}
$$

Because

$$
\frac{\|x^{(k)} - x^{(0)}\|^\beta}{x_s^{(k)} - x_s^{(0)}} = [\frac{\|x^{(k)} - x^{(0)}\|^2}{(x_s^{(k)} - x_s^{(0)})^{2/\beta}}]^{\beta/2} = [\frac{\sum\limits_{i=1}^{n}(x_i^{(k)} - x_i^{(0)})^2}{(x_s^{(k)} - x_s^{(0)})^{2/\beta}}]^{\beta/2}
$$

$$
= [\frac{\sum\limits_{i=1}^{n}(x_i^{(k)} - x_i^{(0)})^2}{(x_s^{(k)} - x_s^{(0)})^2}]^{\beta/2} \frac{1}{(x_s^{(k)} - x_s^{(0)})^{1-\beta}} \leq \frac{n^{\beta/2}}{(x_s^{(k)} - x_s^{(0)})^{1-\beta}},
$$

hence, $\lim\limits_{k\to\infty} \frac{\|x^{(k)} - x^{(0)}\|^\beta}{x_s^{(k)} - x_s^{(0)}} = 0$, we take the limit in both sides of (8), the left-hand side is less then zero, the right-hand side is equal to zero, it is impossible. Thus, $\{x^{(k)}\}$ is a bounded sequence. From the first equation of (1) and H2, it is easy to see that $\{y^{(k)}\}$ is a bounded sequence too. □

Corollary 3.3. *For a given $\omega^{(0)} \in R^n_{++} \times R^n_{++}$, if 0 is a regular value of H and the pair $(M,\ N)$ is a P_* pair, then $\Gamma_{\omega^{(0)}}$ is a bounded curve in $R^n_+ \times R^n_+ \times (0,1]$.*

Corollary 3.4. *For a given $\omega^{(0)} \in R^n_{++} \times R^n_{++}$, if 0 is a regular value of H and the pair $(M,\ N)$ is a P pair, then $\Gamma_{\omega^{(0)}}$ is a bounded curve in $R^n_+ \times R^n_+ \times (0,1]$.*

Theorem 3.5. *Let H be defined by (1), the pair $(M,\ N)$ is a $P(\tau,\alpha,\beta)$ pair, then for almost all $\omega^{(0)} \in R^n_{++} \times R^n_{++}$, the zero-point set $H^{-1}_{\omega^{(0)}}(0)$ of homotopy map (1) contains a smooth curve $\Gamma_{\omega^{(0)}}$, which starts from $(\omega^{(0)},1)$. As $\mu \to 0$, the limit point is $(x^{(*)}, y^{(*)}, 0)$ of $\Gamma_{\omega^{(0)}}$ and $(x^{(*)}, y^{(*)}) \in R^n_+ \times R^n_+$ is a solution of the problem (HLCP).*

Proof. By Lemmas 3.1 and 3.2, we know that $\Gamma_{\omega^{(0)}}$ is a bounded smooth curve. Lemma 2.3 thus implies that $\Gamma_{\omega^{(0)}}$ is diffeomorphic to a unit circle or a unit interval $(0, 1]$. Notice that

$$\left.\frac{\partial H}{\partial \omega}\right|_{\substack{\mu = 1 \\ \omega = \omega^{(0)}}} = \begin{pmatrix} 0 & -N \\ Y^{(0)} & X^{(0)} \end{pmatrix}$$

is nonsingular. That means that $\Gamma_{\omega^{(0)}}$ is not diffeomorphic to a unit circle. Therefore, it is diffeomorphic to the unit interval $(0, 1]$. Let $(\omega^{(*)}, \mu_*)$ be a limit point of $\Gamma_{\omega^{(0)}}$. Only the following four cases are possible:

(1) $\mu_* \in [0, 1], \|(x^{(*)}, y^{(*)})\| \to \infty$;
(2) $\mu_* \in (0, 1), \|(x^{(*)}, y^{(*)})\| < \infty$, there exists $i \in L$, such that $x_i^{(*)} = 0$,or $y_i^{(*)} = 0$;
(3) $\mu_* = 1, \|(x^{(*)}, y^{(*)})\| < \infty$;
(4) $\mu_* = 0, \|(x^{(*)}, y^{(*)})\| < \infty$.

Lemma 3.2 implies that case (1) is impossible. The equation $H(\omega^{(0)}, \omega^{(0)}, 1) = 0$ has only one solution $(\omega^{(0)}, 1) \in R^n_{++} \times R^n_{++} \times (0, 1]$, which means that case (3) is impossible. If case (2) holds, then $y_i^{(*)} = 0$ and $\mu_* \in (0, 1)$, that results in $x_i^{(*)} = \infty$, which is impossible. Thus, case (2) does not hold. So only case (4) holds. □

From Theorem 3.5, we know that for almost all $\omega^{(0)} \in R^n_{++} \times R^n_{++}$, the homotopy equation (1) implicitly defines a smooth curve $\Gamma_{\omega^{(0)}}$, which we call the homotopy path. Let s denote the arc length of $\Gamma_{\omega^{(0)}}$, we can parameterize $\Gamma_{\omega^{(0)}}$ with respect to s in the form of following

$$\begin{gathered} H(\omega(s), \mu(s)) = 0, \\ \omega(0) = \omega^{(0)},\ \mu(0) = 1. \end{gathered} \tag{9}$$

Differentiating the first equation of (9) yields:

Theorem 3.6. *The homotopy path $\Gamma_{\omega^{(0)}}$ is determined by the following system of ordinary differential equations with the given initial* *values*

$$(10)\qquad \begin{aligned} & H'_{\omega^{(0)}}(\omega,\mu)\begin{pmatrix} \dot{\omega}(s) \\ \dot{\mu}(s) \end{pmatrix} = 0, \\ & \|(\dot{\omega}(s),\dot{\mu}(s))\| = 1, \\ & \omega(0) = \omega^{(0)}, \mu(0) = 1, \dot{\mu}(0) < 0, \end{aligned}$$

and the ω*-component of* $(\omega(s^{(*)}),\mu(s^{(*)}))$*, for* $\mu(s^{(*)}) = 0$ *is a solution of (HLCP).*

We discuss how to trace numerically homotopy path $\Gamma_{\omega^{(0)}}$. A standard procedure is the predictor-corrector method which used an explicit difference scheme for solving numerically (10) to give a predictor point and then uses a locally convergent iterative method for solving the nonlinear system of equation (9) to give a corrector point. A simple predictor-corrector procedure algorithm can be found in [11].

We give the following proposition to obtain the positive direction of the predictor-corrector algorithm in [11].

Theorem 3.7. *If* $\Gamma_{\omega^{(0)}}$ *is smooth, then the positive direction* $\eta^{(0)}$ *at the initial point* $\omega^{(0)}$ *satisfies*

$$\operatorname{sign}\det\begin{pmatrix} H'_{\omega^{(0)}}(\omega^{(0)},1) \\ \eta^{(0)^T} \end{pmatrix} = (-1)^{2n+1}\operatorname{sign}\det(N).$$

Proof. From

$$H'_{\omega^{(0)}}(\omega,\mu) = \begin{pmatrix} (1-\mu)M & -N & -(Mx+q)+Ny^{(0)} \\ Y & X & -X^{(0)}y^{(0)} \end{pmatrix},$$

where $Y = diag(y)$. Let

$$\omega = \omega^{(0)}, \mu = 1,$$

we obtain

$$\begin{aligned} H'_{\omega^{(0)}}(\omega^{(0)},1) &= \begin{pmatrix} 0 & -N & -(Mx^{(0)}+q)+Ny^{(0)} \\ Y^0 & X^{(0)} & -X^{(0)}y^{(0)} \end{pmatrix} \\ &= \begin{pmatrix} M_1 & M_2 \end{pmatrix}, \end{aligned}$$

where

$$M_1 \in R^{2n\times 2n}, M_2 \in R^{2n\times 1}.$$

The tangent vector $\xi^{(0)}$ of $\Gamma_{\omega^{(0)}}$ at $(\omega^{(0)}, 1)$ satisfies

$$\begin{pmatrix} M_1 & M_2 \end{pmatrix} \begin{pmatrix} \xi_1^0 \\ \xi_2^0 \end{pmatrix} = 0,$$

where $\xi_1^0 \in R^{2n}, \xi_2^0 \in R$ and $\xi^{(0)} = \begin{pmatrix} \xi_1^{(0)} \\ \xi_2^{(0)} \end{pmatrix}$.

By simple computation, we have $\xi_1^0 = -M_1^{-1} M_2 \xi_2^{(0)}$, so we have

$$\begin{aligned}
\det \begin{pmatrix} H'_{\omega^{(0)}}(\omega^{(0)}, 1) \\ (\xi^0)^T \end{pmatrix} &= \det \begin{pmatrix} M_1 & M_2 \\ \xi_1^{(0)} & \xi_2^{(0)} \end{pmatrix} \\
&= \det \begin{pmatrix} M_1 & M_2 \\ -M_2^T M_1^{-T} & 1 \end{pmatrix} \xi_2^{(0)} \\
&= \det \begin{pmatrix} M_1 & M_2 \\ 0 & 1 + M_2^T M_1^{-T} M_1^{-1} M_2 \end{pmatrix} \xi_2^{(0)} \\
&= \det(M_1) \xi_2^{(0)} (1 + M_2^T M_1^{-T} M_1^{-1} M_2).
\end{aligned}$$

By the definition of M_1, we have

$$\det(M_1) = \det \begin{pmatrix} 0 & -N \\ Y^{(0)} & X^{(0)} \end{pmatrix} = (-1)^{2n} \det(N) \prod_{i=1}^{n} y_i^{(0)}.$$

Hence,

$$\det \begin{pmatrix} H'_{\omega^{(0)}}(\omega^{(0)}, 1) \\ (\xi^0)^T \end{pmatrix}$$

$$= (-1)^{2n} \text{sign} \det(N) \prod_{i=1}^{n} y_i^{(0)} \xi_2^{(0)} (1 + M_2^T M_1^{-T} M_1^{-1} M_2),$$

so we give

$$\operatorname{sign}\det\begin{pmatrix} H'_{\omega^{(0)}}(\omega^{(0)},1) \\ \xi^{(0)^T} \end{pmatrix} = (-1)^{2n+1}\operatorname{sign}\det(N).$$

□

We give some examples to show that the method developed is feasible and effective.

Example 3.1

$$N=\begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 6 \\ 2 & 6 & 9 \end{pmatrix}, M=\begin{pmatrix} 7 & 1 & 0 \\ 17 & 3 & 0 \\ 20 & 4 & 0 \end{pmatrix}, q=\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}^T,$$

$$Mu-Nv=0 \Leftrightarrow \begin{cases} v_1 = u_1 - u_2 \\ v_2 = u_2 + 3u_1 \\ v_3 = 0 \end{cases}$$

Thus, the pair (M,N) is a P_*-pair with $\tau = 3$.

$x_1^{(0)}$	$x_2^{(0)}$	$x_3^{(0)}$	$y_1^{(0)}$	$y_2^{(0)}$	$y_3^{(0)}$	μ_0
1	1	1	1	1	1	1
$x_1^{(*)}$	$x_2^{(*)}$	$x_3^{(*)}$	$y_1^{(*)}$	$y_2^{(*)}$	$y_3^{(*)}$	μ_*
1.0000	0.0000	0.0001	0.0001	2.0001	1.0000	0.0001

TABLE 1. Results of example 3.1

Example 3.2

$$N=\begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 6 \\ 2 & 6 & 9 \end{pmatrix}, M=\begin{pmatrix} -6 & -2 & -2 \\ -15 & -4 & -6 \\ -18 & -4 & -9 \end{pmatrix}, q=\begin{pmatrix} 20 \\ 50 \\ 62 \end{pmatrix},$$

$$Mu-Nv=0 \Leftrightarrow \begin{cases} v_1 = -2u_2 \\ v_2 = -3u_1 \\ v_3 = -u_3 \end{cases}$$

Thus, the pair (M,N) is not a P_*-pair. Indeed, let $u = (1\ 1\ 1)^T$, then $v = (-2\ -3\ -1)^T$.

$x_1^{(0)}$	$x_2^{(0)}$	$x_3^{(0)}$	$y_1^{(0)}$	$y_2^{(0)}$	$y_3^{(0)}$	μ_0
1	1	1	1	1	1	1
$x_1^{(*)}$	$x_2^{(*)}$	$x_3^{(*)}$	$y_1^{(*)}$	$y_2^{(*)}$	$y_3^{(*)}$	μ_*
0.0000	0.0000	0.0000	3.9997	5.9994	1.9999	0.0001

TABLE 2. Results of example 3.2

Example 3.3

$$N = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M = \begin{pmatrix} -6 & -23 & 12 & -13 \\ -5 & -16 & 8 & -6 \\ -3 & -11 & 7 & -3 \\ -2 & -4 & 3 & 1 \end{pmatrix}, q = \begin{pmatrix} 70 \\ 44 \\ 23 \\ 8 \end{pmatrix},$$

$$Mu - Nv = 0 \Leftrightarrow \begin{cases} v_1 = u_1 - 2u_2 + 3u_3 - 4u_4 \\ v_2 = -u_1 + 2u_2 - 3u_3 + u_4 \\ v_3 = u_1 - 3u_2 + u_3 - 5u_4 \\ v_4 = -2u_1 - 4u_2 + 3u_3 + u_4 \end{cases}$$

Thus, the pair (M, N) is not a P_*-pair. Indeed, let $u = (1\ 1\ 1\ 1)^T$, then $v = (-2\ \ -1\ \ -6\ \ -2)^T$.

$x_1^{(0)}$	$x_2^{(0)}$	$x_3^{(0)}$	$x_4^{(0)}$	$y_1^{(0)}$	$y_2^{(0)}$	$y_3^{(0)}$	$y_4^{(0)}$	μ_0
1	1	1	1	1	1	1	1	1
$x_1^{(*)}$	$x_2^{(*)}$	$x_3^{(*)}$	$x_4^{(*)}$	$y_1^{(*)}$	$y_2^{(*)}$	$y_3^{(*)}$	$y_4^{(*)}$	μ_*
0.0000	0.0000	0.0000	0.0000	4.9996	5.9995	6.9993	7.9993	0.0001

TABLE 3. Results of example 3.3

REFERENCES

[1] Sznajder Roman, M. Seetharama Gowda, *Generalizations of P_0-and $P-$Properties; Extended Vertical and Horizontal Linear Complementarity Problems*, Linear Algebra And Its Applications, **223/224** (1995), 695–715.

[2] Jianzhong Zhang, Naihua Xiu, *Global s-type error bound for the extended linear complementarity problem and applications*, Mathematical Programming, **88(2)** (2000), 38–40.

[3] J. Frédéric Bonnans, Florian A. Potra, *Infeasible path following algorithms for linear complementarity problems*, Unité de recherche INRIA Rocquencourt Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France).

[4] R. D. C. Monteiro, T. Tsuchiya, *Limiting behavior of the derivatives of certain trajectories associated with a monotone horizontal linear complementarity problem*, Mathematics of operations research, **21(4)** (1996), 793–814.

[5] Josef Stoer, *High order long-step methods for solving linear complementarity problems*, Annals of Operations Research, **103** (2001), 149–159.

[6] Xing Liu, Florlian A. Potar, *Corrector-predictor meyhods for sufficient linear complementarity problems in a wide neighborhood of the central payh*, SIAM Journal on Optimization, **17** (2006), 871–890.

[7] Filiz Gurtuna, Cosmin Petra, Florian A. Potra, Olena Shevchenko, Adrian Vancea, *Corrector-predictor meyhods for sufficient linear complementarity problems*, Comput. Optim., **148** (2011), 453–485.

[8] Y. B. Zhao. and G., Isac., *Quasi-P_* Maps, $P(\tau, \alpha, \beta)-$ Maps, Exceptional Family of Element, and Complementarity Problems*, Journal Optimization Theory and Applications, **105(1)** (2000), 213–231.

[9] E. L. Allgower, K. Georg, *Numerical continuation methods: An introduction*, Springer-Verlag, Berlin, New York, 1990.

[10] G. L. Naber, *Topological method in Euclidean space*, Cambridge University Press, London, 1980.

[11] Lin Zhenghua, Li Yong, and Yu Bo, *A combined homotopy interior point method for general nonlinear programming problems*, Applied mathematics and computation, **80** (1996), 209–224.